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Breakdown of simple scaling in Abelian sandpile models in one dimension

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We study the Abelian sandpile model on decorated one-dimensional chains. We determine the structure and the asymptotic form of distribution of avalanche sizes in these models, and show that these differ qualitatively from the behavior on a simple linear chain. We find that the probability distribution of the total number of topplings s on a finite system of size L is not described by a simple finite-size scaling form, but by a linear combination of two simple scaling forms $\text{Prob}_L(s)=(1/L)f_1(s/L)+(1/L^2)f_2(s/L^2)$, for large L, where f_1 and f_2 are some scaling functions of one argument.

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In recent years there has been a lot of interest in systems showing self-organized criticality (SOC) [1–3]. However, the precise conditions under which the steady state of a driven system shows critical (long range) correlations are not well understood for nonconservative systems [4–6]. In the case of systems with conservation laws [7,8], for example in sandpile models with local conservation of sand, it is easily shown that the average number of topplings in an avalanche diverges as a power of the system size [9,10]. This, however, is not sufficient to ensure criticality, if criticality is defined as the existence of power law tails in the distribution of avalanche sizes [11].

Lacking a general theory, most studies of SOC depend upon numerical simulations for evidence of criticality. To incorporate the effect of finite-size cutoffs, it is usual to fit data to a finite-size scaling form in which the critical exponents of the infinite system appear as parameters. However, on the basis of extensive numerical studies of one-dimensional sandpile automata, Kadanoff and co-workers [12,13] have argued that there is more than one characteristic length scale in many of these models. Consequently, a simple finite-size scaling does not describe the statistics of avalanches, and a more general "multifractal" scaling form seems necessary.

As the finite-size scaling assumption based on a single scaling form is widely used in the studies of SOC, it seems desirable to test it in a simple analytically tractable model. This we do in this Rapid Communication for a class of one-dimensional Abelian sandpile models (ASM's). We find that, for large L, the distribution functions of duration t of an avalanche, and of the number of distinct sites toppled s_d, in our model do have a simple scaling form. However, the distribution function of total number of topplings s, and of the maximum number of topplings n_c at any one site in the avalanche, does *not* have a simple scaling form, but a more complicated linear combination of two simple scaling forms (LC2SSF)

$$\text{Prob}_L(X)=L^{-\beta_1}f_1(XL^{-\nu_1})+L^{-\beta_2}f_2(XL^{-\nu_2}), \qquad (1)$$

for large L, where $\beta_1=\nu_1=0$ and $\beta_2=\nu_2=1$ for $X=n_c$ and $\beta_1=\nu_1=1$ and $\beta_2=\nu_2=2$ for $X=s$, and f_1 and f_2 are scaling functions, different for $X=s$ and $X=n_c$. We also find that this behavior is quite robust and does not depend on the choice of the unit cell, but in general the functions f_1 and f_2 are not universal.

The ASM on a simple linear chain has been studied earlier by Bak *et al.* [14], and in more detail by Ruelle and Sen [15]. We consider ASM on one-dimensional chains formed by joining a single type of unit cells (see Fig. 1). Such decorated chains are the simplest generalization of the linear chain. We have studied two cases in detail. Case A is a chain of alternating double and single bonds. Case B is a chain of diamonds joined together by single bonds. We solve these models exactly in the large L limit, and find that the avalanche distribution function shows a nontrivial behavior, different from that of the simple linear chain (case C). In fact, the behavior of the ASM in case C is not typical of one-dimensional ASM's.

The model is defined as follows: A site on the chain is denoted by a pair of indices (i,j), where $i=1$ to L labels the unit cell and j numbers a site within the unit cell. In case A, j ranges from 1 to 2, and in case B, from 1 to 4. At each site (i,j) there is an integer variable h_{ij}, called the height of the sandpile at that site. A particle is added at a randomly selected site. If the height h_{ij} is greater than a preassigned threshold height h_{ij}^c at that site it topples, and loses one particle to each of its neighbors. We choose h_{ij}^c to be independent of i and equal to the coordination number of sites of type j. A toppling at a boundary site causes a loss of one particle from the system. The process of toppling continues until there are no unstable sites. After the system is stabilized a new particle is added.

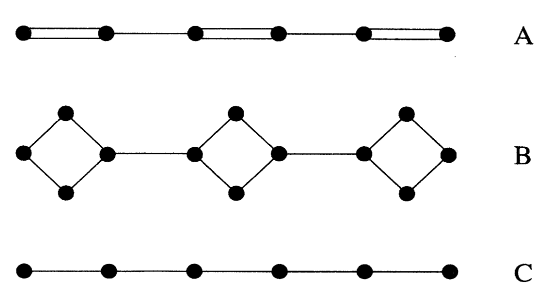

FIG. 1. The one-dimensional chains formed by joining (A) doublets, (B) diamonds, (C) single sites.

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The critical steady state is easy to characterize using the general theory of ASM's [9]. In the steady state all recurrent configurations occur with equal probability. The set of recurrent configurations is characterized by the burning algorithm (see [2], also [16]). In this algorithm, sites are burned (deleted) recursively using the following rule: A site is burned if its height is greater than the number of bonds joining it to its unburned neighbors. A stable configuration is recurrent if and only if all sites are eventually burned.

In this algorithm, the sites can be burned in any order. We choose the convention that the burning starts from the left boundary and continues rightward as long as possible. The unit cell where the rightward burning stops will be called the break point. Afterwards, the burning is allowed to proceed leftwards from the right boundary. It is easy to see that in a recurrent configuration of model A, the allowed values of (h_{i1},h_{i2}), for i on the left of the break point, are (3,3) and (3,2). For i on the right of the break point these are (3,3) and (2,3) and at the break point these are (2,3), (3,1), and (1,3). Since each doublet other than the break point has only two possible configurations, the entropy per site, defined as the logarithm of the total number of recurrent configurations divided by the number of sites, is finite and equals ln(2)/2 in the large L limit. For the simple linear chain, the entropy per site in the SOC state is zero. This fact is responsible for its nongeneric behavior.

To the left (right) of the break point, the left (right) site of a doublet always has height 3, and the probability of right (left) site of a doublet having height 2 and 3 is $\frac{1}{2}$ each. The break point can occur at any of the L doublets with equal probability. Averaging over the position of the break point, this implies that the probabilities of the left site of the ith doublet having height 2 and 3 are $i/(2L)$ and $1-i/(2L)$, respectively. Similarly, the probabilities of the right site of a doublet having height 2 and 3 are $\frac{1}{2}(1-i/L)$ and $\frac{1}{2}(1+i/L)$, respectively. Thus the average height profile in the SOC state varies linearly with i in case A, and the SOC state is *not translationally invariant even far away from the boundaries*. This feature is not present in case C.

Now we describe the spread of avalanches in model A, which again differs qualitatively from case C. Without loss of generality, we may assume that the point where the particle is added (to be called the source site), is to the left of the break point. Then clearly, if the configuration of the doublet left to it is (3,2), the avalanche does not spread to the left and propagates a distance of order L up to the break point on the right. Each site affected by the avalanche topples only once, and the total number of topplings in an avalanche is of order L. Such an avalanche is said to be of type I. The probability of such avalanches is $\frac{1}{4}$. One can easily check that otherwise the avalanche propagates a distance of order L on both sides of the source point. In such avalanches n_c is of order L, and the total number of topplings in an avalanche is of order L^2. Such an avalanche is said to be of type II (see Fig. 2). As the probability that the addition of a particle will cause an avalanche is $\frac{3}{4}$, the fractional number of avalanches of type I is $\frac{1}{3}$.

The probability distributions of total number of topplings s, total number of distinct sites toppled s_d, duration t, and the number of times the source site topples n_c, for type I avalanches can be calculated easily. It is convenient to work

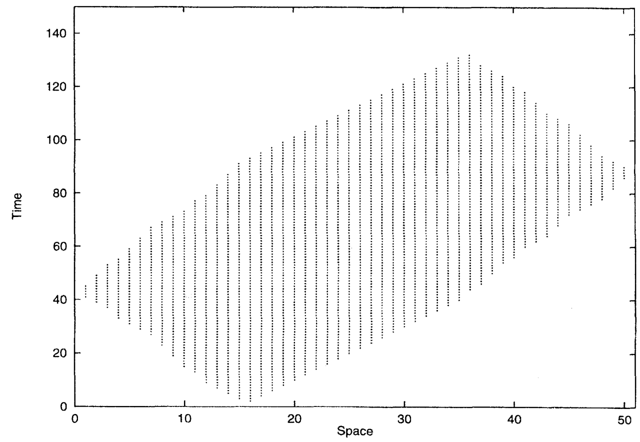

FIG. 2. The evolution of type II avalanche in model A. A dot denotes a toppling event.

with the scaled variable $\alpha\equiv i/L$ and $\beta\equiv j/L$, such that $\alpha,\beta\in[0,1]$, where i and j are the position of source point and break point on the chain, respectively. It can be easily verified that for type I avalanches, s, s_d and t are all equal to $2(\beta-\alpha)L$. Thus the probability distribution of s/L, s_d/L, and t/L for given α and β, and given that an avalanche is of type I, is a δ function at $2(\beta-\alpha)$. Using the fact that α and β are independent random variables uniformly distributed between 0 and 1, averaging over α and β we find for type I avalanches

$$\text{Prob}_L(X|\text{type I})=[1-X/(2L)]/L \quad \text{for } X\leq 2L, \qquad (2)$$

where $X=s$, s_d, or t. In type I avalanches any site topples at most once, so $n_c=1$.

The type II avalanches show a more complicated and interesting structure. The avalanche fronts, i.e., the left and right boundary sites of the active region at any time, do not move uniformly in time; the spreading rate depends on the local height configuration. However, for distances $\gg 1$, one can define an average velocity. The analysis of these avalanches become easy using the decomposition of avalanches into waves of toppling proposed by Ivashkevich *et al.* [17]. In each wave of toppling the source site topples only once and all other sites topple until they are stable. Waves of toppling propagate in exactly the same way as the burning front in the burning algorithm. Thus a unit cell which cannot be fully burned from the left (right) side, stops a left (right) propagating wave coming towards it. (However, in this process, it is itself modified and the next wave may be able to cross it.) We refer to such cells as left (right) stoppers. The stoppers slow down the spreading of avalanches. Obviously the first wave propagates up to the break point with a velocity equal to 1 site per time step. To calculate the velocity towards the left from the source point we note that (a) a doublet of type (3,3) is crossed in 2 time steps and (b) a doublet of type (3,2) is crossed in 4 time steps because it stops the first wave approaching to it and it is crossed in 2 time steps by the next wave which follows after 2 time steps of the previous wave. Thus the average time taken by the avalanche front to cross a doublet is 3, which implies the average velocity is $\frac{2}{3}$ sites per time step. Similarly, one can show that if the avalanche crosses the break point on the right it will advance with an average velocity which is also $\frac{2}{3}$ sites per time steps. The velocity with which an avalanche front recedes backwards after it has hit the boundary is $\frac{2}{3}$. Details will be presented elsewhere.

Since the avalanche front moves with an average velocity, it forms a polygon in the space-time history of the avalanche (see Fig. 2) [18]. The number of sides in the polygon depends on the positions of the source point α and break point β and on whether the break point is crossed by the avalanche or not. For example, if $\beta>\alpha>5\beta/6$ and the break point is not crossed, then the polygon has only four edges. If $1-6\alpha>\beta>\alpha$, and the break point is crossed, then the polygon has six edges. There are seven possible cases of polygons which need to be analyzed separately. Quantities such as s_d, t, and n_c which are proportional to the linear size of the polygon scale as L, but s varies as the area of the polygon and scales as L^2. The expressions of scaled variables s/L^2, s_d/L, t/L, and n_c/L can be easily evaluated in terms of α and β for each case. The probability distribution function for given α and β is a sum of two δ functions corresponding to the cases of whether the break point is crossed by the avalanche or not. Averaging over α and β we find

$$\text{Prob}_L\,(q|\text{type II})=\sum_{i=1,2}\int_0^1\int_0^1 d\alpha\, d\beta\; C_i\delta(q-q_i(\alpha,\beta)), \tag{3}$$

where q is the generic notation for s/L^2, s_d/L, t/L, and n_c/L, $C_1=1/3$ is the probability that a type II avalanche crosses the break point and $C_2=2/3$ is the probability that it does not cross the break point, and q_1 and q_2 denote expressions of q in terms of α and β in the two cases. The full explicit expression is quite complicated and will be presented elsewhere.

However, some of the important features of the distribution function can be understood by simple arguments. Since s_d is the extension of the polygon along the horizontal axis, s_d/L is a linear function of α and β in each of the seven cases. Hence the probability distribution of s_d/L is a piecewise linear function. The same argument works for t and n_c also. The total number of topplings s is proportional to the area of the polygon. Therefore s/L^2 is a quadratic function of α and β in each of the seven cases. The probability distribution in this case is quite complicated and diverges as $(s/L^2)^{-1/2}$ for small (s/L^2).

Using weighted sum of Eqs. (2) and (3), we obtain the full probability distributions. Since n_c and s scale differently for type I and type II avalanches, the distributions of these quantities have the form given in Eq. (1). Other quantities such as s_d and t scale as L for both types of avalanches. Therefore the distributions of s_d and t have a simple scaling form.

The treatment is easily extended to other types of unit cells also. For example, in case B, the unit cell is a diamond. In this case also, an avalanche always spreads up to the break point. The spread of avalanches to the other side will be either of order L or of order 1. Thus, again, there are two types of avalanches. A detailed calculation shows that these occur with relative frequencies 5:8 on the average. While the velocities of avalanche front are different in this case, the

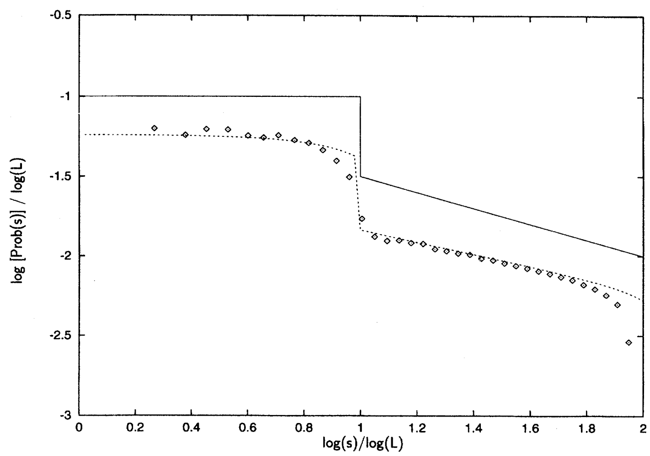

FIG. 3. The log-log plot of Prob(s) vs s. The solid line shows the exact asymptotic behavior for $L\rightarrow\infty$, and the dotted line shows the theoretical curve for $L=100$.

probability distribution functions for both type I and type II avalanches have the same qualitative features irrespective of the velocities. For type I avalanches, $t\sim s_d\sim(\beta-\alpha)L$ to order L. Thus the probability distributions of s_d and t have the same linear form as in model A, but the slopes depend on the velocities. The variable n_c has the probability distribution $\text{Prob}(n_c)\sim 2^{-n_c}$. As $s\sim n_c(\beta-\alpha)L$, this implies that the scaling function f_1 in Eq. (1) is a piecewise linear function with many segments. For type II avalanches the space-time history of active sites forms a polygon exactly as in model A, except that the slopes of the edges of the polygon (velocities of avalanche fronts) are different. Therefore the probability distributions have the same qualitative behavior as in model A. However, the exact forms of functions f_1 and f_2 are not the same in cases A and B, and these functions are *not universal.* In case C, there are no avalanches of type I, and the simple scaling ansatz works [15].

In the multifractal approach, one defines the function $f(\alpha)$ by the relation that an avalanche of size $X=L^{\alpha}$ occurs with a probability which scales as $L^{f(\alpha)}$, for large L. The exponent $f(\alpha)$ defined as $\lim_{L\rightarrow\infty}\log[\text{Prob}_L(X)]/\log(L)$ is a function of the α. For our Abelian model it is easy to see from Eq. (1) that $f(\alpha)$ is a nonincreasing piecewise linear function for $X=s$ (see Fig. 3). We have also shown results of a computer simulation of the model for $L=100$ for 2×10^5 avalanches. Also shown is the theoretical curve using Eq. (1) for $L=100$ (dotted line) and $L=\infty$ (solid line). Clearly there is very good agreement with simulation data. We note that the f versus α curve is qualitatively similar to that obtained in [13], and that the approach to the $L\rightarrow\infty$ limit is quite slow.

As the LC2SSF involves only a finite number of unknown parameters, its use when simple scaling fails is preferable over the more general multifractal form. We also note that we find the breakdown of simple scaling without the appearance of additional new length scales in our model (the only relevant scales are lattice spacing, and the size of system).

Similar behavior may be expected in other effectively one-dimensional models. For example, consider an ASM on a square lattice with a periodic boundary condition in one direction and an open boundary condition in the other direction, forming a cylindrical surface. It is effectively a one-dimensional model for the length of the cylinder L much larger than its width M. We expect three types of avalanches: type I and II, and finite avalanches of size less than M (which do not ring the cylinder and are two dimensional in character). This shows that an LC3SSF would describe this situation. It remains to be seen whether this behavior survives in higher dimensions or is specific to one-dimensional models.

To summarize, we have determined an exact asymptotic finite-size scaling behavior of the distribution of avalanche sizes in the Abelian sandpile model on a class of decorated one-dimensional chains. We find that in these models the SOC state is not translationally invariant, and the probability distribution of s and n_c, unlike the simple linear chain, is described by a linear combination of two simple scaling forms, and not by a simple scaling form.

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